6.3 Constellations for Digital Modulation Schemes

6.3.1 PAM

Definition 6.47. Recall, from 6.7, that **PAM signal waveforms** are represented as

$$s_m(t) = A_m p(t), \quad 1 \le m \le M$$

where $p(t)$ is a pulse and $A_m \in \mathcal{A}$.

$w_1 = s_1 = A_1 p$

$\phi_1 = \frac{w_1}{\sqrt{E_{w_1}}} = \frac{A_1 p}{\sqrt{E_{s_1}}} = \frac{A_1 p}{\sqrt{A_1^2 E_p}} = \frac{p}{\sqrt{E_p}}$

6.48. Clearly, PAM signals are one-dimensional since all are multiples of the same basic signals. We define

$w_2 = s_2 - \text{proj}_{w_1} s_2 = s_2 - \text{proj}_{s_1} s_2$

$= s_2 - \frac{\langle A_2 p, A_1 p \rangle}{\langle A_1 p, A_1 p \rangle} A_1 p$

$= s_2 - A_2 p = 0$

$$\phi(t) = \frac{p(t)}{\sqrt{E_p}}$$

as the basis for the PAM signals above. In which case,

$$s_m(t) = A_m \sqrt{E_p}\,\phi(t), \quad 1 \le m \le M$$

and the corresponding one-dimensional vector representation is

$$\mathbf{s}^{(m)} = A_m \sqrt{E_p}.$$

The corresponding signal space diagrams for $M = 2$, $M = 4$, and $M = 8$ are shown in Figure 18.

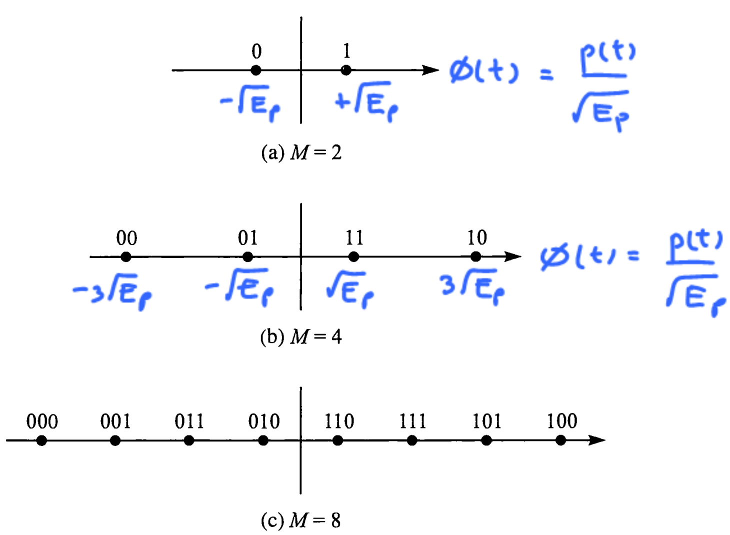

Figure 18: Constellation for PAM signaling

6.49. In **Amplitude-Shift Keying** (ASK), $p(t) = g(t)\cos(2\pi f_c t)$ where f_c is the carrier frequency.

If $p(t) = g(t)\cos(2\pi f_c t + \phi)$, then $E_p = \frac{1}{2}E_g$ ← HW

6.3.2 Phase-Shift Keying (PSK)

Definition 6.50. In **digital phase modulation**, the M signal waveforms are represented as

$$s_m(t) = g(t)\cos\left(2\pi f_c t + \underbrace{\frac{2\pi}{M}(m-1)}_{\theta_m}\right), \quad m = 1, 2, \ldots, M \qquad (36)$$

where

- $g(t)$ is the signal pulse shape and
- $\theta_m = \frac{2\pi}{M}(m-1)$, $m = 1, 2, \ldots, M$ is the M possible phases of the carrier that convey the transmitted information.

Digital phase modulation is usually called **phase-shift keying (PSK)**.

6.51. The PSK signal waveforms defined in (36) have equal energy:

6.52. Note that

(a) From the cos identity

$2\pi f_c t$ $\quad\theta_m$

$\cos x = \frac{1}{2}(e^{jx} + e^{-jx})$

$\sin x =$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$

we have

$= \cos(\theta_m)\big(g(t)\cos\quad\big) + \sin(\theta)\big(-g(t)\sin\quad\big)$

$$s_m(t) = g(t)\cos(\theta_m)\cos(2\pi f_c t) - g(t)\sin(\theta_m)\sin(2\pi f_c t).$$

(b) $\underbrace{g(t)\cos(2\pi f_c t)}_{x(t)}$ and $\underbrace{-g(t)\sin(2\pi f_c t)}_{y(t)}$ are orthogonal. $\langle x, y\rangle = 0$

$x(t) = g(t)\cos(2\pi f_c t) \xrightarrow{\mathcal{F}} X(f) = \frac{1}{2}\Big(\underbrace{G(f-f_c)}_{A} + \underbrace{G(f+f_c)}_{B}\Big)$

$y(t) = -g(t)\sin(2\pi f_c t) \xrightarrow{\mathcal{F}} Y^*(f) = +\frac{1}{2j}\Big(\underbrace{G^*(f-f_c)}_{C} - \underbrace{G^*(f+f_c)}_{D}\Big)$

$\langle x, y\rangle = \langle X, Y\rangle = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df = \frac{1}{4j}\int_{-\infty}^{\infty} \underbrace{(A+B)(C-D)}\,df$ $= AC - AD + BC - BD$

$= \frac{1}{4j}\int_{-\infty}^{\infty} |G(f - \cancel{f_c})|^2 df - \frac{1}{4j}\int_{-\infty}^{\infty} |G(f + \cancel{f_c})|^2 df + 0 + 0$

$= 0$

$G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Assumption: $G(f) = 0$ for $f > f_c$, $f < -f_c$

Therefore, we define

$$\phi_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t), \quad = \frac{g(t)\cos}{\sqrt{E_g/2}} \tag{37}$$

$$\phi_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t). = \frac{-g(t)\sin}{\sqrt{E_g/2}} \tag{38}$$

In which case,

$$s_m(t) = \sqrt{\frac{E_g}{2}} \cos(\theta_m)\, \phi_1(t) + \sqrt{\frac{E_g}{2}} \sin(\theta_m)\, \phi_2(t).$$

Therefore the signal space dimensionality is $N = 2$ and the resulting vector representations are

$$\mathbf{s}^{(m)} = \left(\sqrt{\frac{E_g}{2}} \cos(\theta_m), \sqrt{\frac{E_g}{2}} \sin(\theta_m) \right)^T.$$

6.53. Signal space diagrams for BPSK (binary PSK, $M = 2$), QPSK (quaternary PSK, $M = 4$), and 8-PSK are shown in Figure 19.

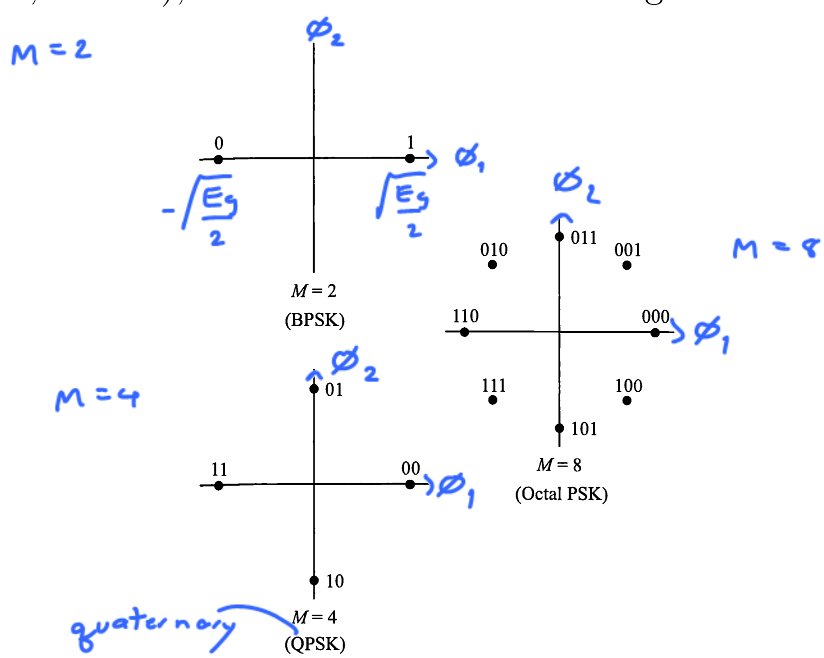

Figure 19: Signal space diagrams for BPSK, QPSK, and 8-PSK.

Note that BPSK corresponds to one-dimensional signals, which are identical to binary PAM signals.

6.3.3 Quadrature Amplitude Modulation (QAM)

Definition 6.54. In **Quadrature Amplitude Modulation (QAM)**, two separate b-bit symbols from the information sequence on two quadrature carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ are transmitted simultaneously. The corresponding signal waveforms may be expressed as

$$s_m(t) = A_m^{(I)} g(t) \cos(2\pi f_c t) - A_m^{(Q)} g(t) \sin(2\pi f_c t), \quad m = 1, 2, \ldots, M \tag{39}$$

where

- $A_m^{(I)}$ and $A_m^{(Q)}$ are the information-bearing signal amplitudes of the quadrature carriers and
- $g(t)$ is the signal pulse.

Equivalently,

$$\begin{aligned} s_m(t) &= \mathrm{Re}\left\{\left(A_m^{(I)} + jA_m^{(Q)}\right) g(t)\, e^{j2\pi f_c t}\right\} && (40)\\ &= \mathrm{Re}\left\{r_m e^{j\theta_m} g(t)\, e^{j2\pi f_c t}\right\} && (41)\\ &= r_m g(t) \cos(2\pi f_c t + \theta_m) && (42)\end{aligned}$$

where

- $r_m = \sqrt{\left(A_m^{(I)}\right)^2 + \left(A_m^{(Q)}\right)^2}$ is the magnitude and
- θ_m is the argument or phase

of the complex number $A_m^{(I)} + jA_m^{(Q)}$.

6.55. From (42), it is apparent that the QAM signal waveforms may be viewed as combined amplitude (r_m) and phase (θ_m) modulation. In fact, we may select any combination of M_1-level PAM and M_2-phase PSK to construct an $M = M_1 M_2$ combined **PAM-PSK signal constellation**.

- If $M_1 = 2^{b_1}$ and $M_2 = 2^{b_2}$, the combined PAM-PSK signal constellation results in the simultaneous transmission of $b_1 + b_2 = \log_2 M_1 M_2$ binary digits occurring at a symbol rate $R/(b_1 + b_2)$.

6.56. From (39), it can be seen that, similar to the PSK case, $\phi_1(t)$ and $\phi_2(t)$ given in (37) and (38) can be used as an orthonormal basis for QAM signals. The dimensionality of the signal space for QAM is $N = 2$. Using this basis, we have

$$s_m(t) = A_m^{(I)}\sqrt{\frac{E_g}{2}}\phi_1(t) + A_m^{(Q)}\sqrt{\frac{E_g}{2}}\phi_2(t)$$

which results in vector representations of the form

$$\mathbf{s}^{(m)} = \left(A_m^{(I)}\sqrt{\frac{E_g}{2}}, A_m^{(Q)}\sqrt{\frac{E_g}{2}}\right)^T.$$

6.57. Examples of signal space diagrams for combined PAM-PSK are shown in Figure 20, for $M = 8$ and $M = 16$.

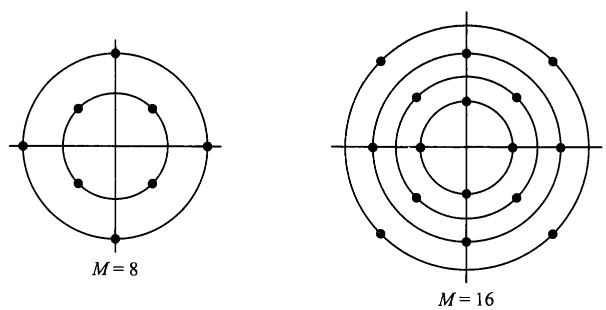

Figure 20: Examples of combined PAM-PSK constellations.

In the special case where the signal amplitudes are taken from the set of discrete values $\mathcal{A} = \{(2m - 1 - M), m = 1, 2, \ldots, M\}$, the signal space diagram is rectangular, as shown in Figure 21.

6.58. PAM and PSK can be considered as special cases of QAM. In QAM signaling, both amplitude and phase carry information, whereas in PAM and PSK only amplitude or phase carries the information.

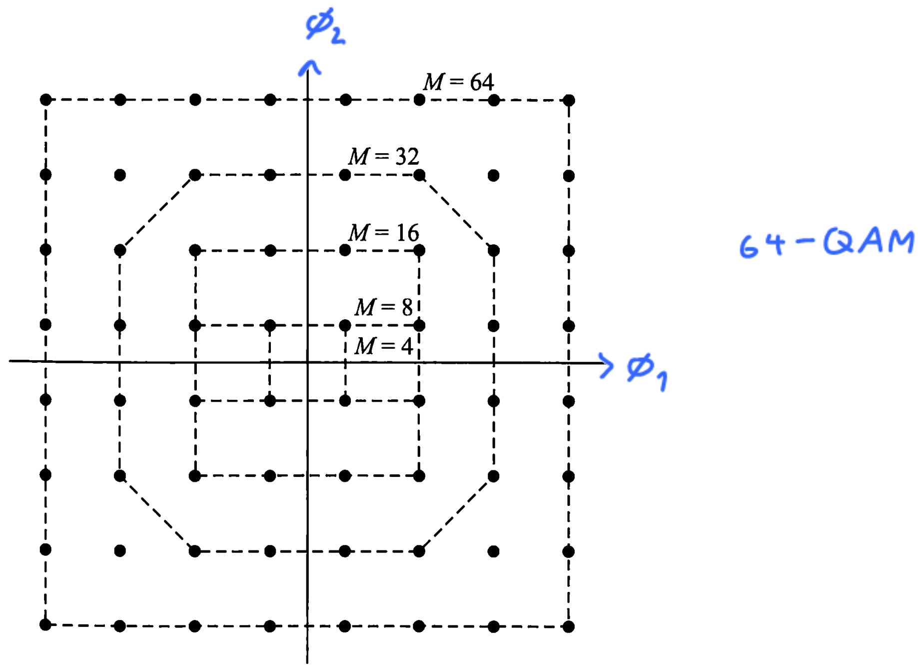

Figure 21: Several signal space diagrams for rectangular QAM.

6.3.4 Orthogonal Signaling

Definition 6.59. In orthogonal signaling, the waveforms $s_m(t)$ are orthogonal and of equal energy E. In which case, the orthonormal set $\{\phi_m(t), 1 \le m \le N\}$ defined by

$$\phi_m(t) = \frac{s_m(t)}{\sqrt{E}}, \quad 1 \le m \le M$$

can be used as an orthonormal basis for representation of $\{s_m(t), 1 \le m \le M\}$. The resulting vector representation of the signals will be

$$\begin{aligned} s^{(1)} &= \left(\sqrt{E}, 0, 0, \ldots, 0\right), \\ s^{(2)} &= \left(0, \sqrt{E}, 0, \ldots, 0\right), \\ \vdots &= \quad \vdots \\ s^{(M)} &= \left(0, 0, 0, \ldots, \sqrt{E}\right). \end{aligned}$$